Chapter 1

WAKES AND IMPEDANCES

1.1 Wake Fields

A positively charged particle at rest has static electric field going out radially in all directions. In motion with velocity v, magnetic field is generated. As the particle velocity approaches c, the velocity of light, the electric and magnetic fields are pancake-like, the electric field is radial and magnetic field azimuthal (the Liénard-Wiechert fields) with an open angle of about $1/\gamma$, where $\gamma = \sqrt{1 - v^2/c^2}$. It is interesting to point out that no matter how far away, this pancake is always perpendicular to the path of motion. In other words, the fields move with the test particle without any lagging behind as illustrated in Fig. 1.1. Such a field pattern is, of course, the steady-state solution of the problem.

When placed inside a perfectly conducting beam pipe, the pancake of fields is trimmed by the beam pipe. A ring of negative charges will be formed on the walls of the beam pipe where the electric field ends, and these image charges will travel at the same pace with the particle, creating the so-called *image current*. If the wall of the beam pipe is not perfectly conducting or contains discontinuities, the movement of the image charges will be slowed down, thus leaving electromagnetic fields behind. For example, when coming across a cavity, the image current will flow into the walls of the cavity, exciting fields trapped inside the cavity. These fields left behind by the particle are called *wake fields*, which are important because they influence the motion of the particles that follow.

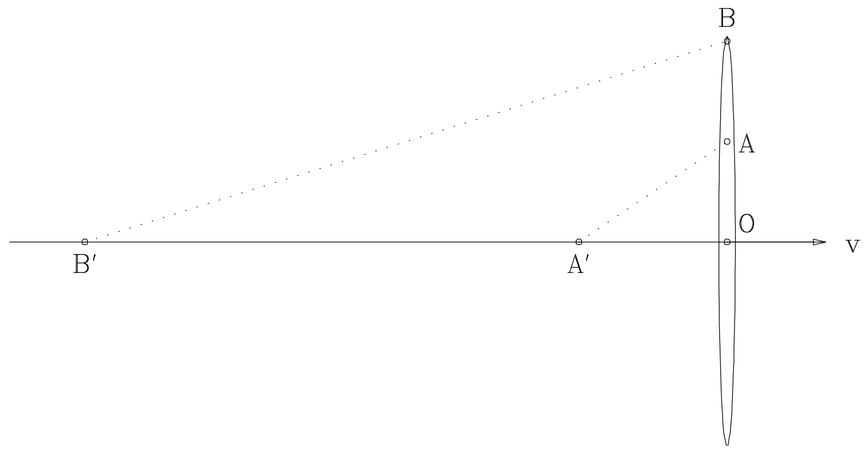

Figure 1.1: Schematic drawing of pan-cake electromagnetic fields emitted by an ultra-relativistic particle traveling with velocity v. The pan-cake is always perpendicular to the path of the particle and travels in pace with the particle no matter how far away the fields are from the particle. There is no violation of causality because fields at points A and B come from the particle at different locations. Fields from A are from A' at a time OA'/v ago, while fields at B from point B' at a time OB'/v ago.

In addition to the wake fields, the electromagnetic fields seen by the beam particle consist of also the external fields from the magnets, rf, etc. The electric field $\vec{E}$ and magnetic flux density $\vec{B}$ can be written as

$$(\vec{E},\vec{B})_{\substack{\text{seen by}\\ \text{particles}}} = (\vec{E},\vec{B})_{\substack{\text{external, from}\\ \text{magnets, rf, etc.}}} + (\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}} \tag{1.1}$$

where

$$(\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}} \left\{ \begin{array}{ll} \propto & \text{beam intensity} \\ \ll & (\vec{E},\vec{B})_{\text{external}} \ . \end{array} \right.$$

Note that the last restriction, which is certainly not true in plasma physics, allows wake fields to be treated as perturbation. This perturbation, however, will break down when potential-well distortion is large. In that case, the potential-well distortion has to be included into the non-perturbative part. What we need to compute are the wake fields at a distance z behind the source particle and their effects on the test or witness particles that make up the beam. The computation of the wake fields is nontrivial. So approximations are required.

1.2 Two Appromixations

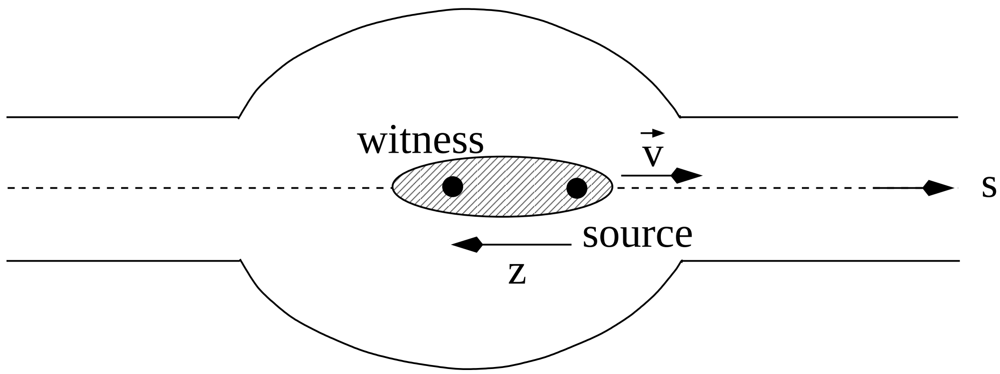

Figure 1.2: Schematic drawing of of a witness particle at a distance z behind the some source particle in a beam. Both particles are traveling along the direction s with velocity $\vec{v}$.

At high energies, the particle beam is rigid and the following two approximations apply:*
(1) **The rigid beam approximation**, which says that the beam traverses the discontinuity of the vacuum chamber rigidly and the wake field perturbation does not affect the motion of the beam during the traversal of the discontinuity. This is a good approximation even in the presence of synchrotron oscillations, because the longitudinal distance between two beam particles changes negligibly in a revolution turn relative to the circumference of the accelerator ring. This implies that the distance z of the test particle behind some source particle as shown in Fig. 1.2 does not change.
(2) **The impulse approximation**. Although the test particle carrying a charge q sees a wake force $\vec{F}$ coming from $(\vec{E}, \vec{B})$, what it cares is the impulse

$$\Delta\vec{p} = \int_{-\infty}^{\infty} dt\ \vec{F} = \int_{-\infty}^{\infty} dt\ q(\vec{E} + \vec{v} \times B) \tag{1.2}$$

as it completes the traversal through the discontinuity at its fixed velocity $\vec{v}$. Note that MKS units have been used in Eq. (1.2) and will be adopted throughout the rest of the lectures. We will therefore be coming across the electric permitivity of free space $\epsilon_0 = 10^7/(4\pi c^2)$ farads/m and the magnetic permeability of free space $\mu_0 =$

*This approach to the Panofsky-Wenzel Theorem was presented by A.W. Chao at the OCPA Accelerator School, Hsinchu, Taiwan, August 3-12, 1998.

$4\pi \times 10^{-7}$ henry/m. These two quantities are related to the free-space impedance Z_0 and velocity of light c by

$$
\begin{aligned}
Z_0 &= \sqrt{\frac{\mu_0}{\epsilon_0}} = 2.99792458 \times 40\pi = 376.730313 \text{ Ohms} \ , \\
c &= \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792458 \times 10^8 \text{ m/s} \ .
\end{aligned}
\tag{1.3}
$$

Both $\vec{E}$, $\vec{B}$, and $\vec{F}$ are difficult to compute even at high beam energies. However, the impulse $\Delta\vec{p}$ has great simplifying properties through the Panofsky-Wenzel (P-W) theorem, which forms the basis of wake potentials and impedances.

1.3 Panofsky-Wenzel Theorem

Maxwell equations for a particle in the beam are:

$$
\left\{
\begin{array}{ll}
\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0} & \text{Gauss's law for electric charge,} \\
\vec{\nabla}\times\vec{B} - \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} = \mu_0\beta c\rho\hat{s} & \text{Ampere's law,} \\
\vec{\nabla}\cdot\vec{B} = 0 & \text{Gauss's law for magnetic charge,} \\
\vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0 & \text{Faraday's \& Lenz law.}
\end{array}
\right.
\tag{1.4}
$$

We have replaced the current density with $\vec{j} = \beta c\rho\hat{s}$ where ρ is the charge density of the beam. The beam particle velocity $|\vec{v}| = \beta c$ will be treated as a constant, which is the result of the rigid-beam approximation, and is certainly true at high energies when $\beta \approx 1$. Note that we have been denoting the s-axis as the direction of motion of the beam, while reserving z as the distance the witness particle is *ahead* the source particle. For a circular ring, the s-axis constitutes the axis of symmetry of the vacuum chamber. Together with the transverse coordinates x and y, they form an instantaneous right-handed Cartesian coordinate system. Thus, the above wake fields $\vec{E}$ and $\vec{B}$ as well as wake force $\vec{F}$ are function of x, y, s, t. From the rigid beam approximation, the location of the test particle, s, is not independent, but is related to t by $s = z + \beta ct$, where z is regarded as time-independent and the location of the source particle is given by $s_{\text{source}} = \beta ct$. Since we are looking at the field behind a source, z is negative.

The Lorentz force on the test particle of charge q is $\vec{F} = q(\vec{E} + \beta c\hat{s} \times \vec{B})$. Here the rigid-beam approximation has also been used by requiring that the test particle has the same velocity as all other beam particles. It follows that

$$\vec{\nabla}\cdot\vec{F} = \frac{q\rho}{\epsilon_0\gamma^2} - \frac{q\beta}{c}\frac{\partial E_s}{\partial t} \ , \tag{1.5}$$

$$\vec{\nabla}\times\vec{F} = -q\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial s}\right)\vec{B} \ . \tag{1.6}$$

We are only interested in the impulse

$$\Delta\vec{p}(x,y,z) = \int_{-\infty}^{\infty} dt \ \vec{F}(x,y,z+\beta ct,t) \ ; \tag{1.7}$$

i.e., the integration of $\vec{F}$ along a rigid path with z being held fixed. Applying the curl to both sides,

$$\underset{\substack{\uparrow \\ \text{this } \vec{\nabla} \text{ refers} \\ \text{to } x,y,z}}{\vec{\nabla}}\times\Delta\vec{p}(x,y,z) = \int_{-\infty}^{\infty} dt \ \left[\underset{\substack{\uparrow \\ \text{this } \vec{\nabla} \text{ refers} \\ \text{to } x,y,s}}{\vec{\nabla}}\times\vec{F}(x,y,s,t)\right]_{s=z+\beta ct} \ , \tag{1.8}$$

we obtain for the right side,

$$\begin{aligned} \text{Right Side} &= -q\int_{-\infty}^{\infty} dt \ \left[\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial s}\right)\vec{B}(x,y,s,t)\right]_{s=z+\beta ct} \\ &= -q\int_{-\infty}^{\infty} dt \ \frac{d\vec{B}}{dt} = -q\vec{B}(x,y,z+\beta ct,t)\Big|_{t=-\infty}^{\infty} = 0 \ . \end{aligned} \tag{1.9}$$

We therefore arrive at relation

$$\vec{\nabla}\times\Delta\vec{p} = 0 \ , \tag{1.10}$$

which is the P-W theorem. It is important to note that so far no boundary conditions have been imposed. The P-W theorem is valid for any boundaries! The only needed inputs are the two approximations: the rigid-bunch approximation and the impulse approximation. The P-W theorem even does not require $\beta = 1$. It just requires $\beta \approx 1$ so that β can remain constant. Thus, the P-W theorem is very general.

The P-W theorem can be decomposed into a component parallel to the $\hat{s}$ and one perpendicular to $\hat{s}$. The decomposition is obtained by taking dot product and cross product of $\hat{s}$ with Eq. (1.10):

$$\vec{\nabla}\cdot(\hat{s}\times\Delta\vec{p}) = 0 \ , \tag{1.11}$$

$$\frac{\partial}{\partial z}\,\Delta\vec{p}_\perp = \vec{\nabla}_\perp \Delta p_s \; . \tag{1.12}$$

Equation (1.11) says something about the transverse components of $\Delta\vec{p}$, which becomes, in Cartesian coordinates,

$$\frac{\partial \Delta p_x}{\partial y} = \frac{\partial \Delta p_y}{\partial x} \; . \tag{1.13}$$

On the other hand, Eq. (1.12) relates $\Delta\vec{p}_\perp$ and $\Delta\vec{p}_z$, that the transverse gradient of the longitudinal impulse is equal to the longitudinal gradient of the transverse impulse. Thus, the P-W theorem strongly constraints the components of $\Delta\vec{p}$.

There is an important supplement to the P-W theorem, which states:

$$\beta = 1 \longrightarrow \vec{\nabla}_\perp\!\cdot\!\Delta\vec{p}_\perp = 0 \; . \tag{1.14}$$

Proof:

$$\begin{aligned}
\vec{\nabla}\!\cdot\!\Delta\vec{p} \;&= \int_{-\infty}^{\infty} dt\left[\vec{\nabla}\!\cdot\!\vec{F}(x,y,s,t)\right]_{s=z+ct} = -\frac{q}{c}\int_{-\infty}^{\infty} dt\left[\frac{\partial E_s}{\partial t}\right]_{s=z+ct} \\
&= q\int_{-\infty}^{\infty} dt\left[\frac{\partial E_s}{\partial s}\right]_{s=z+ct} = \frac{\partial}{\partial z}\Delta p_s \; ,
\end{aligned}$$

where we have used the fact that the longitudinal component of the wake force is independent of the magnetic flux density. For the second last step, use has been made of

$$\frac{\partial}{\partial t}E_s(s,t) = \frac{d}{dt}E_s(s,t) - \frac{ds}{dt}\frac{\partial}{\partial s}E_s(s,t) \; . \tag{1.15}$$

It is important to note that $4\pi q\rho/\gamma^2$, the space charge term of $\vec{\nabla}\cdot\vec{F}$ in Eq. (1.6) has been omitted because $\beta = 1$.

1.4 Cylindrically Symmetric Chamber

When the beam of cylindrical cross section is inside a cylindrically symmetric vacuum chamber, naturally cylindrical coordinates will be used. Some differential operators in the cylindrical coordinates are listed in Table 1.16. The P-W theorem, Eq. (1.10), and the supplemental theorem, Eq. (1.14), become [2]

Table 1.1: Differential operators in the cylindrical coordinates. Here $\vec{A}$ is a vector and Φ is a scalar.

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_s}{\partial s} \ ,$$

$$\vec{\nabla}\times\vec{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial s} - \frac{\partial A_\theta}{\partial s}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r}\right) + \hat{s}\left(\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right) \ ,$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial s^2} \ .$$

$$\left\{\begin{array}{rcl} \dfrac{\partial}{\partial r}(r\Delta p_\theta) & = & \dfrac{\partial}{\partial\theta}\Delta p_r \ , \\ \dfrac{\partial}{\partial z}\Delta p_r & = & \dfrac{\partial}{\partial r}\Delta p_s \ , \\ \dfrac{\partial}{\partial z}\Delta p_\theta & = & \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\Delta p_s \ , \\ \dfrac{\partial}{\partial r}(r\Delta p_r) & = & -\dfrac{\partial}{\partial\theta}\Delta p_\theta \qquad (\beta=1) \ . \end{array}\right. \tag{1.16}$$

Now, this set equations for $\Delta\vec{p}$ becomes surprisingly simple. It does not contain any source terms and is completely independent of boundaries, which can be conductors, resistive wall, dielectric, or even plasma. This result solely arises from the Maxwell equations plus the two approximations.

There is no loss of generality by letting $\Delta p_z \sim \cos m\theta$ with $m \geq 0$. Then, we get

$$\Delta p_s = \Delta\tilde{p}_s\cos m\theta \quad \longrightarrow \quad \Delta p_r = \Delta\tilde{p}_r\cos m\theta \quad \text{and} \quad \Delta p_\theta = \Delta\tilde{p}_\theta\sin m\theta \ , \tag{1.17}$$

where $\Delta\tilde{p}_s$, $\Delta\tilde{p}_r$, and $\Delta\tilde{p}_\theta$ are θ-independent. The set of equations for $\Delta\vec{p}$ becomes

$$\left\{\begin{array}{rcl} \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_\theta) & = & -m\Delta\tilde{p}_r \ , \\ \dfrac{\partial}{\partial z}\Delta\tilde{p}_r & = & \dfrac{\partial}{\partial r}\Delta\tilde{p}_s \ , \\ \dfrac{\partial}{\partial z}\Delta\tilde{p}_\theta & = & -\dfrac{m}{r}\Delta\tilde{p}_s \ , \\ \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_r) & = & -m\Delta\tilde{p}_\theta \qquad (\beta=1) \ . \end{array}\right. \tag{1.18}$$

From the first and last equations, we must have, for $m=0$,

$$\Delta\tilde{p}_\theta = 0 \qquad \text{and} \qquad \Delta\tilde{p}_\theta = 0 \ , \tag{1.19}$$

otherwise they will be proportional to r^{-1} which is singular at $r=0$. From the same two equations, we get, for $m \neq 0$,

$$\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(r\Delta\tilde{p}_r)\right] = m^2\Delta\tilde{p}_r \ , \tag{1.20}$$

and therefore

$$\Delta p_r(r,\theta,z) \sim r^{m-1}\cos m\theta \ . \tag{1.21}$$

Now the whole solution can be written as, for all $m \geq 0$,

$$\left\{\begin{array}{l} v\Delta\vec{p}_\perp = -qI_mW_m(z)mr^{m-1}\left(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right) \ , \\ \\ v\Delta p_s = -qI_mW'_m(z)r^m\cos m\theta \ . \end{array}\right. \tag{1.22}$$

In above, $W_m(z)$ is called the *transverse wake function of azimuthal* m and $W'_m(z)$ the *longitudinal wake function of azimuthal* m. They are related because of the P-W theorem. The wake functions are functions of one variable z only, and are the only remaining unknown. They must be solved with boundary conditions. Recall that the complicated Maxwell-Vlasov equation that involves $\vec{E}$, $\vec{B}$, and sources has been reduced drastically to solving just for W_m.

More comments about Eq. (1.22) are in order. The original solution in the top line of Eq. (1.22) was for $m \neq 0$ only. However, we can always define a $W_0(z)$ which is the anti-derivative of $W'_0(z)$ so that the solution holds for all m. Although $W_0(z)$ has no physical meaning, yet it will be helpful in discussions below. In Eq. (1.22), q is the charge of the test particle and I_m is the electric mth multipole of the source particle. For a source particle of charge e at an offset a from the axis of the cylindrical beam pipe, $I_m = ea^m$. Thus, W'_m has the dimension of force per charge square per length$^{(2m-1)}$ or Volts/Coulomb/m^{2m}, while W_m has the dimension of force per charge square per length^{2m} or Volts/Coulomb/m^{2m-1}. The negative signs on the right sides arise just from a convention. For example, we want the longitudinal wake $W'm(z)$ to be positive when the impulse acting on the test particle is decelerating.

Recall that we have been looking at the wake force on a particle traveling at $s=z+vt$ behind a source particle traveling at $s=vt$. Thus $z<0$. When $v \to c$, causality has to be imposed that $W_m(z) = 0$ when $z > 0$. For our discussions below, we will continue to use v instead of c in most places, because we would like to derive stability conditions and growth rates also for machines that are not ultra-relativistic. However, strict causality will be imposed as if the velocity is c.

Immediately behind a source particle, the test particle should receive a retarding force, otherwise a particle will continue to gain energy as it is traveling down the vacuum chamber in direction violation of the conservation of energy. This implies that $W'_m(z) > 0$ when $|z|$ is small, recalling that the $W'_m(z)$ is defined in Eq. (1.22) with a negative sign on the right side. This is illustrated in Fig. 1.3. It will be proved later in Chapter 7.5 that

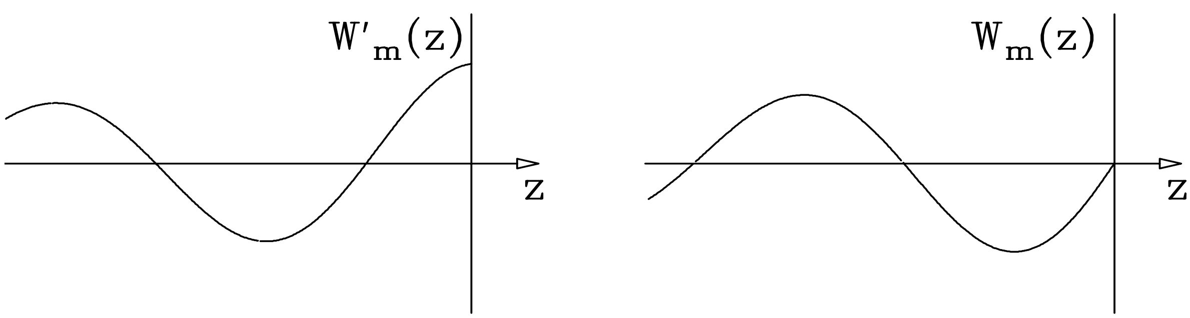

Figure 1.3: The longitudinal wake $W'_m(z)$ vanishes when $z > 0$ and is positive definite when $|z|$ is small. The transverse wake $W_m(z)$ starts out from zero and goes negative as $|z|$ increases.

a particle sees half of its own wake. For the transverse wake $W_m(z)$, it starts out from zero† and goes negative as $|z|$ increases, as required by the P-W theorem. Thus, when the source particle is deflected, a transverse wake force is created in the direction that it will deflect particles immediately following in the *same* direction of the deflection of the source. Again, special attention should be paid to the negative sign on the right side of the definition of $W_m(z)$ in Eq. (1.22). The transverse wake W_m vanishes at $z = 0$ implies that a particle will not see its own transverse wake at all. This leads to the important conclusion that a shorter bunch will be preferred if the transverse wake dominates, and a longer bunch will be preferred if the longitudinal wake dominates.

When $m = 0$ or the monopole, we have $\Delta p_\perp = 0$ while Δp_s is independent of (r, θ) and depends only on z. Thus, particles in a thin transverse slice of the beam will see

†Although it can not be proved that $W_m(0) = 0$, however, most wakes do have this property.

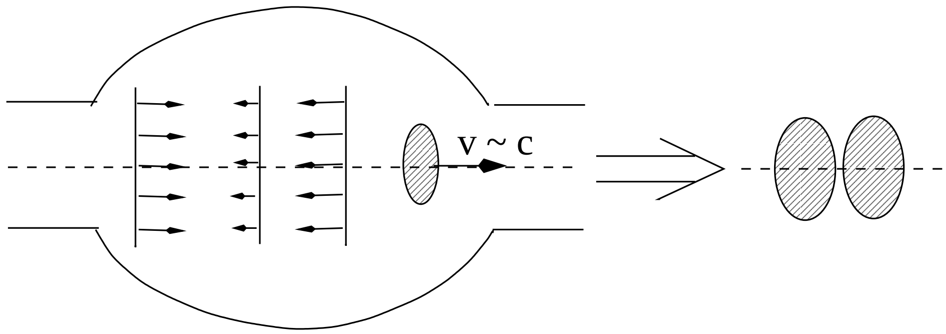

Figure 1.4: All particles in a vertical slice of the beam see exactly the same monopole wake impulse ($m = 0$) from the source according to the slice position z behind the source. This longitudinal variation of impulse effect on the slices can lead to longitudinal microwave instability.

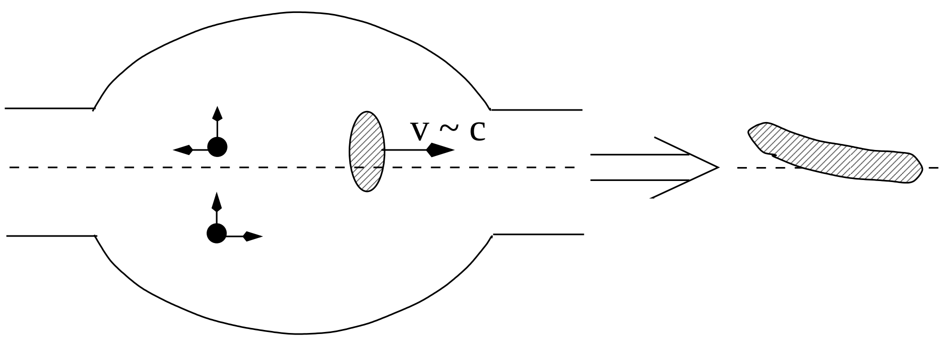

Figure 1.5: Kicks for all the particles in the slice from the dipole wake impulse also have the same magnitude; however, the longitudinal kicks point to forward or backward direction depending on whether the particles are above or below the axis of symmetry.

the same impulse in the s-direction according to the dependence of W_0' on z, as shown in Fig. 1.4. This impulse can lead to self-bunching or microwave instability.

For $m = 1$, we have from Eq. (1.22) that $\Delta p_\perp$ is independent of (r, θ) but depends on z only. All particles in a vertical slice of the beam suffer exactly the same vertical kick from the dipole wake impulse ($m = 1$) which depends only on how far the slice is behind the dipole source, as is shown in Fig. 1.5. Such an impulse can lead to the tilting of the tail of the bunch into a banana shape; it can also cause beam breakup. On the other hand, the dipole longitudinal impulse Δp_s ($m = 1$) is proportional to the offset in the x-direction.

For the sake of convenience, many authors do not like to work with a negative z for the particles that are following. There is another convention that $W_m(z) = 0$ when $z < 0$. This does not change the physics and the direction of the wake forces will not be changed. Thus, instead of Fig. 1.3, we have Fig. 1.6 instead. A price has to be paid

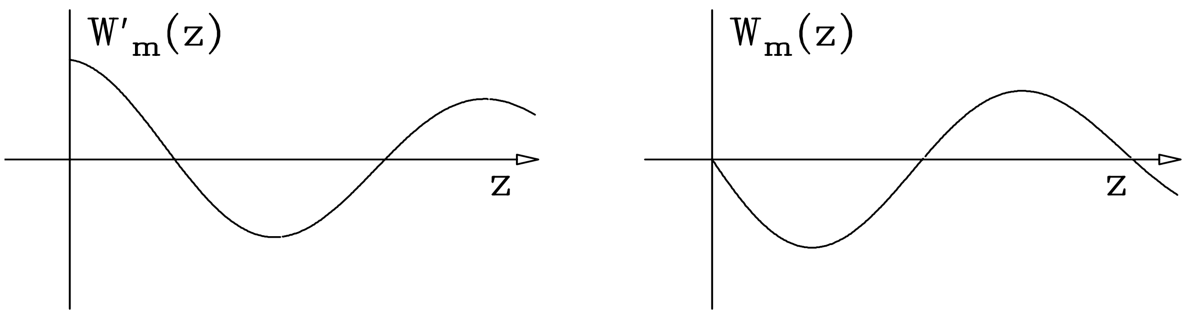

Figure 1.6: This is a different convention that the wake functions $W_m(z)$ vanish when $z < 0$. Since the physics is the same, the wake functions are the same as in Fig. 1.3 and just the direction of z has been changed. In this convention, the interpretation $W_m'(z) \equiv -\frac{d}{dz}W_m(z)$ is required.

for this convention. We must interpret the connection between the longitudinal and transverse wakes as

$$W_m'(z) \equiv -\frac{d}{dz}W_m(z) \ . \tag{1.23}$$

This convention will be used for the rest of the lectures.‡ Fortunately, we will not be using Eq. (1.23) much below, because most longitudinal instabilities are driven dominantly by the monopole longitudinal wake W_0' and most transverse instabilities are driven dominantly by the dipole transverse wake W_1.

‡The readers should be aware of yet another convention in the literature that the wake functions $W_m(z)$ and $W_m'(z)$ are defined in Eq. (1.22) *without* the negative signs on the right sides. As a result, the wake functions will have just the opposite signs of what are depicted in Fig. 1.6.

1.5 Coupling Impedances

Beam particles form a current, of which the component with frequency $\omega/(2\pi)$ is[§] $I(s,t) = \hat{I}e^{-i\omega(t-s/v)}$, where $\hat{I}$ may be complex. This current component at location s and time t will be affected by the wake of the preceding beam particles that pass the point s at time $t-z/v$ with the charge element $I(s,t-z/v)dz/v$. The total *accelerating* voltage seen (or energy gained per unit test charge) will be

$$V(s,t) = -\int_{-\infty}^{\infty} \hat{I}e^{-i\omega[t-(s+z)/v]}W_0'(z)\frac{dz}{v} = -I(s,t)\int_{-\infty}^{\infty} e^{i\omega z/v}W_0'(z)\frac{dz}{v} \ . \tag{1.24}$$

Thus, we can identify the *longitudinal coupling impedance* of the vacuum chamber as

$$Z_0^{\|}(\omega) = \int_{-\infty}^{\infty} e^{i\omega z/v}W_0'(z)\frac{dz}{v} \ . \tag{1.25}$$

This definition is the same as the ordinary impedance in a circuit. However, we have here much more than in a circuit because the current distribution can possess higher multiples.

When the current is displaced transversely by a from the axis of symmetry of the beam pipe, the *deflecting* transverse force acting on a current particle is obtained by summing the charge element $I(s,t-z/v)dz/v$ passing s at time $t-z/v$,

$$\langle F_1^{\perp}(s,t)\rangle = -\frac{qa}{\ell}\int_{-\infty}^{\infty} \hat{I}e^{-i\omega[t-(s+z)/v]}W_1(z)\frac{dz}{v} = -\frac{qa}{\ell}I(s,t)\int_{-\infty}^{\infty} e^{i\omega z/v}W_1(z)\frac{dz}{v} \ , \tag{1.26}$$

where $\langle F_1^{\perp}(s,t)\rangle$ is the transverse force averaged over a length ℓ covering the discontinuity of the vacuum chamber, and is therefore equal to $v\Delta p_\perp/\ell$, with $\Delta p_\perp$ being the transverse impulse studied in the previous sections. For an accelerator ring or storage ring, this length is taken to be the ring circumference C. We identify the *transverse coupling impedance* of the vacuum chamber as

$$Z_1^{\perp}(\omega) = \frac{i}{\beta}\int_{-\infty}^{\infty} e^{i\omega z/v}W_1(z)\frac{dz}{v} \ . \tag{1.27}$$

[§]We are going to use the physicist convention (except in Chapter 7.5) of denoting the frequency dependence by $e^{-i\omega t}$, which leads to the results that the capacitive impedance is positive imaginary while the inductive impedance is negative imaginary. The opposite is true in the engineering convention of $e^{j\omega t}$.

In both Eqs. (1.24) and (1.26), the lower limits of integration have been extended to $-\infty$, because the wake functions vanish when $z < 0$. From Eq. (1.26), it is evident that we can also compute the transverse impedance by integrating the wake force averaged over one turn according to

$$Z_1^{\perp}(\omega) = -\frac{i}{q\beta I_0 a}\int_0^C F_1^{\perp}(s,t)\,ds\ , \tag{1.28}$$

where Ia represents the dipole source current. Since $\mathcal{R}e\,Z_1^{\perp}(\omega) > 0$ implies an energy loss, the force leads the displacement Ia by $\frac{\pi}{2}$, and hence the factor $-i$ in Eq. (1.28). The Lorentz factor $\beta = v/c$ is a convention.

Inversely, the wake functions can be written in terms of the impedances:

$$W_m(z) = -\frac{i\beta}{2\pi}\int_{-\infty}^{\infty} Z_m^{\perp}(\omega)e^{-i\omega z/v}d\omega\ , \tag{1.29}$$

$$W_m'(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_m^{\parallel}(\omega)e^{-i\omega z/v}d\omega\ , \tag{1.30}$$

where the path of integration in both cases is above all the singularities of the impedances so as to guarantee causality.

Note that the longitudinal impedance is mostly the monopole ($m = 0$) impedance and the transverse impedance is mostly the dipole ($m = 1$) impedance, if the beam pipe cross section is close to circular and the particle path is close to the pipe axis. They have the dimensions of Ohms and Ohms/length, respectively. The impedances have the following properties:

1. $Z_0^{\parallel}(-\omega) = [Z_0^{\parallel}(\omega)]^*$ and $Z_1^{\perp}(-\omega) = -[Z_1^{\perp}(\omega)]^*$. (1.31)

2. $Z_0^{\parallel}(\omega)$ and $Z_1^{\perp}(\omega)$ are analytic with poles only in the lower half ω-plane.¶ (1.32)

3. $Z_m^{\parallel}(\omega) = \dfrac{\omega}{c}Z_m^{\perp}(\omega)$, (1.33)

   for cylindrical geometry and each azimuthal harmonic including‖ $m = 0$.

4. $\mathcal{R}e\,Z_0^{\parallel}(\omega) \geq 0$ and $\mathcal{R}e\,Z_1^{\perp}(\omega) \geq 0$ when $\omega > 0$, (1.34)

   if the beam pipe has the same entrance cross section and exit cross section.

5. $\displaystyle\int_0^{\infty} d\omega\ \mathcal{I}m\,Z_m^{\perp}(\omega) = 0$, and $\displaystyle\int_0^{\infty} d\omega\,\frac{\mathcal{I}m\,Z_m^{\parallel}(\omega)}{\omega} = 0$. (1.35)

The first follows because the wake functions are real, the second from the causality of the wake functions, and the third from the Panofsky-Wenzel theorem [1] between transverse and longitudinal electromagnetic forces. $\mathcal{R}e\, Z_m^{\parallel}(\omega) \geq 0$ is the result of the fact that the total energy of a particle or a bunch cannot be increased after passing through a section of the vacuum chamber where there is no accelerating external forces, while $\mathcal{R}e\, Z_m^{\perp}(\omega) \geq 0$ when $\omega > 0$ follows from the Panofsky-Wenzel theorem. The fifth property follows from the assumption that $W_m(0) = 0$.

For a pure resistance R, the longitudinal wake is $W_0'(z) = R\delta(z/v)$. At low frequencies, the wall of the beam pipe is inductive. This wake function is $W_0'(z) = L\delta'(z/v)$, where L is the inductance.

For a nonrelativistic beam of radius a inside a circular beam pipe of radius b, the longitudinal space charge impedance for $m = 0$ is**

$$Z_0^{\parallel}(\omega) = i\frac{\omega}{\omega_0}\frac{Z_0}{2\gamma^2\beta}\left(1 + 2\ln\frac{b}{a}\right) \ , \tag{1.36}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\ \Omega$ is the impedance of free space, μ_0 and ϵ_0 are, respectively, the magnetic permeability and electric permitivity of free space, $\omega_0/(2\pi)$ is the revolution frequency of the beam particle with Lorentz factors γ and β. Although this impedance is capacitive, however, it appears in the form of a negative inductance. The corresponding wake function is

$$W_0'(z) = -\delta'(z/v)\frac{1}{\omega_0}\frac{Z_0}{2\gamma^2\beta}\left(1 + 2\ln\frac{b}{a}\right) \ . \tag{1.37}$$

The $m = 1$ transverse space charge impedance for a length ℓ of the circular beam pipe is

$$Z_1^{\perp}(\omega) = i\frac{Z_0\ell}{2\pi\gamma^2\beta^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right] \ , \tag{1.38}$$

and the corresponding transverse wake function is

$$W_1(z) = \frac{Z_0 c\ell}{2\pi\gamma^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right]\delta(z) \ . \tag{1.39}$$

An important impedance is that of a resonant cavity. Near the resonant frequency $\omega_r/(2\pi)$, the mth multipole longitudinal impedances can be derived from a RLC-parallel

**This expression will be derived in Chapter 3. Here, the space charge force is seen by beam particles at the beam axis. If the force is averaged over the cross section of the beam with a uniform transverse cross section, the first term in the brackets becomes $\frac{1}{2}$ instead of 1.

circuit:

$$Z_m^{\parallel}(\omega) = \frac{R_{ms}}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)} \ , \tag{1.40}$$

where the resonant angular frequency is $\omega = (L_m C_m)^{-1/2}$ and quality factor is $Q = R_{ms}\sqrt{C_m/L_m}$. Here, for the mth multipole, the shunt impedance R_{ms} is in Ohms/m^{2m}, the inductance in henry/m^{2m}, and the capacitance in farad-m^{2m}. The transverse impedance can now be obtained from the P-W theorem of Eq. (1.33):

$$Z_m^{\perp}(\omega) = \frac{c}{\omega}\frac{R_{ms}}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)} \ . \tag{1.41}$$

Another example is the longitudinal impedance for a length ℓ of the resistive beam pipe:

$$Z_0^{\parallel}(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\frac{\ell}{2\pi b\sigma_c\delta_{\rm skin}} \ , \tag{1.42}$$

where b is the radius of the cylindrical beam pipe, σ_c is the conductivity of the pipe wall,

$$\delta_{\rm skin} = \sqrt{\frac{2c}{Z_0\mu_r\sigma_c|\omega|}} \ , \tag{1.43}$$

is the skin depth at frequency $\omega/(2\pi)$, and μ_r is the relative magnetic permeability of the pipe wall. The transverse impedance is

$$Z_1^{\perp}(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\frac{\ell c}{\pi\omega b^3\sigma_c\delta_{\rm skin}} \ , \tag{1.44}$$

and is related to the longitudinal impedance by

$$Z_1^{\perp}(\omega) = \frac{2c}{b^2\omega}Z_0^{\parallel}(\omega) \ . \tag{1.45}$$

The above relation has been used very often to estimate the transverse impedance from the longitudinal. However, we should be aware that this relation holds only for resistive impedances of a cylindrical beam pipe. The monopole longitudinal impedance and the dipole transverse impedance belong to different azimuthals; therefore they should not be related. An example that violates Eq. (1.45) is the longitudinal and transverse space charge impedances stated in Eqs. (1.36) and (1.38).

More expressions for impedances resulting from various types of discontinuity in the vacuum chamber are reprinted from the *Handbook of Accelerator Physics and Engineering* [3] in the following pages.

3.2.5 Explicit Expressions of Impedances and Wake Functions

K.Y. Ng, FNAL

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General Remarks and Notations:

W_m' denotes mth azimuthal longitudinal wake function as a function of distance z for $z<0$. When $z>0$, $W_m'(z)=0$ and $W_m'(0)=\lim_{z\to 0-} W_m'(z)$. Similar for transverse wake W_m.

The mth azimuthal longitudinal impedance $Z_m^\parallel(\omega) = \int e^{-i\omega z/v} W_m^\parallel(z)dz/v$ is related to the transverse impedance of the same azimuthal $Z_m^\perp(\omega) = \int e^{-i\omega z/v} W_m^\perp(z)idz/(\beta v)$ by $Z_m^\parallel = (\omega/c)Z_m^\perp$ (valid when $m \neq 0$). In many cases, $\beta = v/c$ has been set to 1.

Unless otherwise stated, round beam pipe of radius b is assumed. $C = 2\pi R$ is the ring circumference and n is the revolution harmonic. $Z_0 \approx 377\ \Omega$ is the free-space impedance. ϵ_0 and μ_0 are the free-space dielectric constant and magnetic permeability.

Description	Impedances	Wakes
Space-charge: [1] beam radius a in a length L of perfectly conducting beam pipe of radius b.	$\frac{Z_0^\parallel}{n} = i\frac{Z_0 L}{2C\beta\gamma^2}\left[1+2\ln\frac{b}{a}\right]$ $Z_{m\neq 0}^\perp = i\frac{Z_0 L}{2\pi\beta^2\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]$	$W_0' = \frac{Z_0 cL}{4\pi\gamma^2}\left[1+2\ln\frac{b}{a}\right]\delta'(z)$ $W_{m\neq 0} = \frac{Z_0 cL}{2\pi\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]\delta(z)$
Resistive Wall: [1] pipe length L, wall thickness t, conductivity σ_c, skin depth $\delta_{\rm skin}$.	$\frac{Z_m^\parallel}{L} = \frac{\omega}{c}\frac{Z_m^\perp}{L} = \frac{Z_0 c/(\pi b^{2m})}{[1+\mathrm{sgn}(\omega)i](1+\delta_{m0})bc\sqrt{\frac{\sigma_c Z_0 c}{2\vert\omega\vert}} - \frac{ib^2\omega}{m+1} + \frac{imc^2}{\omega}}$ $t \gg \delta_{\rm skin} = \sqrt{2c/(\vert\omega\vert Z_0\sigma_c)}, \quad \vert\omega\vert \gg c\chi/b, \quad \chi = 1/(Z_0\sigma_c b)$	
For $t \gg \delta_{\rm skin}$ and $b/\chi \gg \vert z\vert \approx c/\vert\omega\vert \gg b\chi^{1/3}$.	$Z_m^\parallel = \frac{\omega}{c}Z_m^\perp$ $Z_m^\parallel = \frac{1-\mathrm{sgn}(\omega)i}{1+\delta_{0m}}\frac{L}{\pi\sigma_c\delta_{\rm skin}b^{2m+1}}$	$W_m = -\frac{c}{\pi b^{2m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{\vert z\vert^{1/2}}$ $W_m' = -\frac{c}{2\pi b^{2m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{\vert z\vert^{3/2}}$
For $t \ll \delta_{\rm skin}$ or very low freq., and $b/\chi \gg \vert z\vert \approx c/\vert\omega\vert \gg \sqrt{bt}$.	$\frac{Z_0^\parallel}{L} = -\frac{iZ_0 t\omega}{2\pi bc}, \quad \frac{Z_1^\perp}{L} = -\frac{iZ_0 t}{\pi b^3}$	$\frac{W_0'}{L} = -\frac{Z_0 tc}{2\pi b}\delta'(z), \ \frac{W_1}{L} = -\frac{Z_0 tc}{\pi b^3}\delta(z)$
A *pair* of strip-line BPM's: [2] length L, angle each subtending to pipe axis ϕ_0, forming transmission lines of characteristic impedance Z_c with pipe.	$Z_0^\parallel = 2Z_c\left[\frac{\phi_0}{2\pi}\right]^2\left[2\sin^2\frac{\omega L}{c} - i\sin\frac{2\omega L}{c}\right]$ $Z_1^\perp = \left[\frac{Z_0^\parallel}{\omega}\right]_{\rm pair}\frac{c}{b^2}\left[\frac{4}{\phi_0}\right]^2\sin^2\frac{\phi_0}{2}$	$W_0' = 2Z_c\, c\left[\frac{\phi_0}{2\pi}\right]^2[\delta(z)-\delta(z+2L)]$ $W_1 = \frac{8Z_c\, c}{\pi^2 b^2}\sin^2\frac{\phi_0}{2}[H(z)-H(z+2L)]$
	The strip-lines are assumed to terminate with impedance Z_c at the upstream end.	
Heifets inductive impedance: [3] low freq. pure inductance $\mathcal{L}$. $Z_0^\parallel$ rolls off as $\omega^{-1/2}$.	$Z_0^\parallel = -\frac{i\omega\mathcal{L}}{(1-i\omega a/c)^{3/2}}$ $\longrightarrow -i\omega\mathcal{L}$ as $a\to 0$	$W_0' = \frac{c^2\mathcal{L}}{a\sqrt{\pi a z}}\left[1+\frac{2z}{a}\right]e^{z/a}$ $\longrightarrow c^2\mathcal{L}\delta'(z)$ as $a\to 0$
Pill-box cavity at low freq.: length g, radial depth $h+b$, where $g \le h \ll b$ [6].	$Z_0^\parallel = -i\frac{\omega Z_0}{2\pi cb}\left[gh-\frac{g^2}{2\pi}\right]$ $Z_1^\perp = -i\frac{Z_0}{\pi b^3}\left[gh-\frac{g^2}{2\pi}\right]$	$W_0' = -\frac{Z_0 c}{2\pi b}\left[gh-\frac{g^2}{2\pi}\right]\delta'(z)$ $W_1 = -\frac{Z_0 c}{\pi b^3}\left[gh-\frac{g^2}{2\pi}\right]\delta(z)$

Description	Impedances	Wakes
Pill-box cavity at low freq.: length g, radial depth $h+b$, where $h \ll g \ll b$ [6].	$Z_0^\parallel = -i\dfrac{\omega Z_0 h^2}{\pi^2 cb}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]$ $Z_1^\perp = -i\dfrac{2Z_0 h^2}{\pi^2 b^3}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]$	$W_0' = -\dfrac{Z_0 ch^2}{\pi^2 b}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]\delta'(z)$ $W_1 = -\dfrac{2Z_0 ch^2}{\pi^2 b^3}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]\delta(z)$
Pill-box cavity: length g, radial depth d. At freq. $\omega \gg c/b$, diffraction model applies [1].	$Z_m^\parallel = \dfrac{[1+\mathrm{sgn}(\omega)i]Z_0}{(1+\delta_{m0})\pi^{3/2}b^{2m+1}}\sqrt{\dfrac{cg}{\lvert\omega\rvert}}$ $Z_m^\parallel = \dfrac{\omega}{c}Z_m^\perp$	$W_m = -\dfrac{2Z_0 c\sqrt{2g}}{(1+\delta_{m0})\pi^2 b^{2m+1}}\lvert z\rvert^{1/2}$ $W_m' = \dfrac{Z_0 c\sqrt{2g}}{(1+\delta_{m0})\pi^2 b^{2m+1}}\lvert z\rvert^{-1/2}$
Optical model: [7] A series of cavities of periodic length L. Each cavity has width g, high Q resonances of freq. $\omega_n/(2\pi)$ and loss factor $k_n^{(m)}$ for azimuthal mode m.	$\mathrm{Re}Z_m^\parallel = \sum_{n=1}^{N}\pi k_n^{(m)}\delta(\omega-\omega_n) + \dfrac{2\pi C_{\text{SV}}G(\bar\nu)F(\nu)}{(1+\delta_{m0})b^{2m}}H(\omega-\omega_N)$ $W_m' = \sum_{n=1}^{N}2k_n^{(m)}\cos\dfrac{\omega_n z}{c} + \dfrac{2C_{\text{SV}}G(\bar\nu)}{(1+\delta_{m0})b^{2m}}\int_{\omega_N}^{\infty}d\omega F(\nu)\cos\dfrac{\omega z}{c}$ where $C_{\text{SV}} = 2Z_0 j_{m1}^2/(\pi^2\zeta^2\beta) \approx 650\ \Omega$ for $m=0$ and $1650\ \Omega$ for $m=1$, j_{m1} is first zero of Bessel function J_m, $\zeta = 0.8237$. $G(\bar\nu)=\bar\nu^2 K_1^2(\bar\nu),\ \ F(\nu)=\dfrac{\sqrt{\nu}+1}{(\nu+2\sqrt{\nu}+2)^2},\ \bar\nu=\dfrac{\omega b}{\beta\gamma c},\ \nu=\dfrac{\omega}{\omega_{\text{SV}}}=\dfrac{4b^2\omega}{\zeta^2 c\sqrt{gL}}$	
Formulas for computation of W_m'. $\mathrm{erfc}(x)$ is the complementary error function.	$\int_{\hat\omega}^{\infty}d\omega F(\nu)\cos\dfrac{\omega z}{c} = \omega_{\text{SV}}\tilde F_0(z/c) - \int_0^{\hat\omega}d\omega F(\nu)\cos\dfrac{\omega z}{c}$ $\tilde F_0(x) = \int_0^\infty d\omega F(\nu)\cos\omega x = \dfrac{\pi}{4}(1+4x)e^{2x}\mathrm{erfc}(\sqrt{2x}) - \sqrt{\dfrac{\pi x}{2}}$	
Resonator model for the mth azimuthal, with shunt imp. $R_s^{(m)}$, resonant freq. $\omega_r/(2\pi)$, quality factor Q [1].	$Z_m^\parallel = \dfrac{R_s^{(m)}}{1+iQ\,(\omega_r/\omega-\omega/\omega_r)}$ $Z_m^\perp = \dfrac{c}{\omega}\,\dfrac{R_s^{(m)}}{1+iQ\,(\omega_r/\omega-\omega/\omega_r)}$	$W_m = \dfrac{R_s^{(m)}c\,\omega_r}{Q\bar\omega_r}\,e^{\alpha z/c}\sin\dfrac{\bar\omega_r z}{c}$ where $\alpha = \omega_r/(2Q)$ $\bar\omega_r = \sqrt{\lvert\omega_r^2-\alpha^2\rvert}$
Res. freq. $\omega_{mnp}/(2\pi)$ and shunt impedance $(R_s)_{mnp}$ of a pill-box cavity for nth radial and pth longitudinal modes. Radial depth d and length g. x_{mn} is nth zero of Bessel function J_m [8].	$\dfrac{\omega_{mnp}^2}{c^2} = \dfrac{x_{mn}^2}{d^2}+\dfrac{p^2\pi^2}{g^2}$ $\left[\dfrac{R_s}{Q}\right]_{0np} = \dfrac{Z_0}{x_{0n}^2 J_0'^2(x_{0n})}\dfrac{8c}{\pi g\omega_{0np}}\begin{cases}\sin^2\dfrac{g\omega_{0np}}{2\beta c}\times\dfrac{1}{1+\delta_{0p}} & p \text{ even}\\ \cos^2\dfrac{g\omega_{0np}}{2\beta c} & p \text{ odd}\end{cases}$ $\left[\dfrac{R_s}{Q}\right]_{1np} = \dfrac{Z_0}{J_1'^2(x_{1n})}\dfrac{2c^2}{\pi g d^2\omega_{1np}^2}\begin{cases}\sin^2\dfrac{g\omega_{1np}}{2\beta c} & p\neq 1 \text{ and even}\\ \cos^2\dfrac{g\omega_{1np}}{2\beta c} & p \text{ odd}\end{cases}$	

<table>
<tr><th>Description</th><th>Impedances</th><th>Wakes</th></tr>
<tr><td rowspan="2">Low-freq. response of a <u>pill-box cavity</u>: [4] length g, radial depth d. When $g \gg 2(d-b)$, replace g by $(d-b)$. Here, $S = d/b$.</td><td>$$\frac{Z_0^{\parallel}}{n} = -i\frac{Z_0 g}{2\pi R}\ln S$$ $$Z_1^{\perp} = -i\frac{Z_0 g}{\pi b^2}\frac{S^2-1}{S^2+1}$$</td><td>$$W_0' = -\frac{Z_0 c g}{2\pi}\ln S\,\delta'(z)$$ $$W_1 = -\frac{Z_0 c g}{\pi b^2}\frac{S^2-1}{S^2+1}\,\delta(z)$$</td></tr>
<tr><td colspan="2">Effect will be one half for a step in the beam pipe from radius b to radius d, or vice versa, when $g \gg 2(d-b)$.</td></tr>
<tr><td><u>Iris</u> of half elliptical cross section at low freq.: width $2a$, maximum protruding length h [5].</td><td>$$Z_0^{\parallel} = -i\frac{\omega Z_0 h^2}{4cb}$$ $$Z_1^{\perp} = -i\frac{Z_0 h^2}{2b^3}$$</td><td>$$W_0' = -\frac{Z_0 c h^2}{4b}\,\delta'(z)$$ $$W_1 = -\frac{Z_0 c h^2}{2b^3}\,\delta(z)$$</td></tr>
<tr><td><u>Pipe transition</u> at low freq.: tapering angle θ, transition height h. γ is Euler's constant and ψ is the psi-function [6].</td><td colspan="2">$$Z_0^{\parallel} = \frac{\omega b^2 Z_1^{\perp}}{2c} = -i\frac{\omega Z_0 h^2}{2\pi^2 cb}\left\{\ln\left[\frac{b\theta}{h} - 2\theta\cot\theta\right] + \frac{3}{2} - \gamma - \psi\left(\frac{\theta}{\pi}\right) - \frac{\pi}{2}\cot\theta - \frac{\pi}{2\theta}\right\}$$ $$W_0' = -\left|\frac{Z_0^{\parallel}}{\omega}\right| c^2\delta'(z)\,, \quad W_1 = -\left|Z_1^{\perp}\right| c\delta(z)\,, \quad h\cot\theta \ll b$$</td></tr>
<tr><td><u>Pipe transition</u> at <u>low frequencies</u> with transition height $h \ll b$ [6].</td><td colspan="2">$$Z_0^{\parallel} = \frac{\omega b^2}{2c} Z_1^{\perp} = -i\frac{\omega Z_0 h^2}{2\pi^2 cb}\left(\ln\frac{2\pi b}{h} + \frac{1}{2}\right)$$ $$W_0' = -\left|\frac{Z_0^{\parallel}}{\omega}\right| c^2\delta'(z)\,, \quad W_1 = -\left|Z_1^{\perp}\right| c\delta(z)$$</td></tr>
<tr><td rowspan="2"><u>Kicker</u> with window-frame magnet [9]: width a, height b, length L, beam offset x_0 horizontally, and all image current carried by conducting current plates.</td><td>$$Z_0^{\parallel} = \frac{\omega^2\mu_0^2 L^2 x_0^2}{4a^2 Z_k}$$ $$Z_1^{\perp} = \frac{c\omega\mu_0^2 L^2}{4a^2 Z_k}$$</td><td>$$W_0' = -\frac{c^3\mu_0^2 L^2 x_0^2}{4a^2 Z_k}\,\delta_0''(z)$$ $$W_1 = -\frac{c^3\mu_0^2 L^2}{4a^2 Z_k}\,\delta'(z)$$</td></tr>
<tr><td colspan="2">$Z_k = -i\omega\mathcal{L} + Z_g$ with $\mathcal{L} \approx \mu_0 bL/a$ the inductance of the windings and Z_g the impedance of the generator and the cable. If the kicker is of C-type magnet, x_0 in $Z_0^{\parallel}$ should be replaced by $(x_0 + b)$.</td></tr>
<tr><td><u>Traveling-wave kicker</u> with characteristic impedance Z_c for the cable, and a window magnet of width a, height b, and length L [9].</td><td colspan="2">$$Z_0^{\parallel} = \frac{Z_c}{4}\left[2\sin^2\frac{\theta}{2} - i(\theta - \sin\theta)\right]\,, \quad Z_1^{\perp} = \frac{Z_c L}{4ab}\left[\frac{1-\cos\theta}{\theta} - i\left(1 - \frac{\sin\theta}{\theta}\right)\right]$$ $$W_0' = \frac{Z_c c}{4}\left[\delta(z) - \delta\left(z - \frac{Lc}{v}\right) - \frac{Lc}{v}\delta'(z)\right]$$ $$W_1 = \frac{Z_c v}{4ab}\left[H(z) - H\left(z - \frac{Lc}{v}\right) - \frac{Lc}{v}\delta(z)\right]$$ $\theta = \omega L/v$ denotes the electrical length of the kicker windings and $v = Z_c ac/(Z_0 b)$ is the matched transmission-line phase velocity of the capacitance-loaded windings.</td></tr>
<tr><td><u>Bethe's electric</u> and <u>magnetic moments</u> of a hole of radius a in beam pipe wall [10].</td><td colspan="2">Electric and magnetic dipole moments when wavelength $\gg a$: $\quad \vec{d} = -\frac{2\epsilon_0}{3}a^3\vec{E}\,,\ \vec{m} = -\frac{4}{3\mu_0}a^3\vec{B}$
$\vec{E}$ and $\vec{B}$ are electric and magnetic flux density at hole when hole is absent. This is a diffraction solution for a thin-wall pipe.</td></tr>
</table>

Description	Impedances	Wakes
Small obstacle [5, 11] on beam pipe, size $\ll$ pipe radius, freq. below cutoff. α_e and α_m are electric polarizability and magnetic susceptibility of the obstacle.	$Z_0^{\parallel} = -i\frac{\omega Z_0}{c}\frac{\alpha_e+\alpha_m}{4\pi^2 b^2}$ $Z_1^{\perp} = -i\frac{Z_0(\alpha_e+\alpha_m)}{\pi^2 b^4}\cos\Delta\varphi$	$W_0' = -Z_0 c\frac{\alpha_e+\alpha_m}{4\pi^2 b^2}\,\delta'(z)$ $W_1 = -Z_0 c\frac{\alpha_e+\alpha_m}{\pi^2 b^4}\cos\Delta\varphi\,\delta(z)$
	$\Delta\varphi$ is the azimuthal angle between the obstacle and the direction concerning $Z_1^{\perp}$ and W_1.	
Polarizabilities for various geometry: beam pipe radius is b and wall thickness is t.		
Elliptical hole: major and minor radii are a and d. $K(m)$ and $E(m)$ are complete elliptical functions of the first and second kind, with $m = 1–m_1$ and $m_1 = (d/a)^2$. For long ellipse $\perp$ beam, major axis $a \ll b$, beam pipe radius, because the curvature of the beam pipe has been neglected here [12].	$\alpha_e+\alpha_m=\begin{cases}\frac{\pi a^3 m_1^2[K(m)-E(m))]}{3E(m)[E(m)-m_1K(m)]}\\ \frac{\pi a^3[E(m)-m_1K(m)]}{3[K(m)-E(m)]}\end{cases}\xRightarrow[\text{long ellipse}]{m\to 1}\begin{cases}\frac{\pi d^4[\ln(4a/d)-1]}{3a} & \parallel \text{ beam } d\ll b\\ \frac{\pi a^3}{3\,[\ln(4a/d)-1]} & \perp \text{ beam } a\ll b\end{cases}$ $\alpha_e+\alpha_m \xRightarrow[m\to 0]{\text{circular}} \frac{2a^3}{3}$ circular hole $a=d\ll b$ Above are for $t\ll a$, $\times 0.56$ (circular) or $\times 0.59$ (long ellipse) when $t\geq a$. For higher frequency correction, add to $\alpha_e+\alpha_m$ the extra term, $+\frac{2\pi a^3}{3}\left[\frac{11\omega^2 a^2}{30c^2}\right]$ circular, $\begin{cases}-\frac{\pi a d^2}{3}\left[\frac{\omega^2 a^2}{5c^2}\right] & \parallel \text{ beam long ellipse}\\ +\frac{2\pi a^3}{3}\left[\frac{2\omega^2 a^2}{5c^2[\ln(4a/d)-1]}\right] & \perp \text{ beam long ellipse}\end{cases}$	
Rectangular slot: length L, width w.	$\alpha_e+\alpha_m = w^3(0.1814-0.0344w/L)\quad t\ll a,\quad \times 0.59$ when $t\geq a$	
Rounded-end slot: length L, width w.	$\alpha_e+\alpha_m = w^3(0.1334-0.0500w/L)\quad t\ll a,\quad \times 0.59$ when $t\geq a$	
Annular-ring-shaped cut: inner and outer radii a and $d=a+w$ with $w\ll d$.	$\alpha_e+\alpha_m = \frac{\pi^2 d^2 a}{2\ln(32d/w)-4} - \frac{\pi^2 w^2(a+d)}{16}\quad t\ll d$ $\alpha_e+\alpha_m = \pi d^2 w - \frac{1}{2}w^2(a+d)\quad t\geq d$	
Half ellipsoidal protrusion with semi axes h radially, a longitudinally, and d azimuthally. ${}_2F_1$ is the hypergeometric function.	$\alpha_e+\alpha_m = 2\pi a h d\left[\frac{1}{I_b}+\frac{1}{I_c-3}\right]$ $I_b = {}_2F_1\Big(1,1;\frac{5}{2};1-\frac{h^2}{a^2}\Big)\,,\quad I_c = {}_2F_1\Big(1,\frac{1}{2};\frac{5}{2};1-\frac{a^2}{h^2}\Big)\,,\quad$ if $a=d$ $\alpha_e+\alpha_m = \pi a^3$ if $a=d=h$, $\frac{2\pi h^3}{3[\ln(2h/a)-1]}$ if $a=d\ll h$ $\alpha_e+\alpha_m = \frac{8h^3}{3}\left[1+\left(\frac{4}{\pi}-\frac{\pi}{4}\right)\frac{a}{h}\right]$ if $a\ll h=d$ $\alpha_e+\alpha_m = \frac{8\pi h^4}{3a}\left[\ln\frac{2a}{h}-1\right]$ if $a\gg h=d$	

Array of pill-boxes, box spacing L, each with gap width g, beam pipe radius b. Gluckstern-Yokoya-Bane formula [15] at high freq. to order $(kg)^{-1}$:	For each cavity of length L with $k=\omega/c$, $$Z_0^{\parallel}=\frac{iZ_0L}{\pi kb^2}\left\{1+[1+i\,\mathrm{sgn}(k)]\frac{\alpha L}{b}\sqrt{\frac{\pi}{\lvert k\rvert g}}\right\}^{-1}$$ with $k=\omega/c$. $\alpha=1$ when $g/L\ll 1$ and $\alpha=\alpha_1=0.4648$ when $g/L=1$, the limiting case of infinitely thin irises. In general, with $\Upsilon=g/L$, $\alpha(\Upsilon)=1-\alpha_1\Upsilon^{1/2}-(1-2\alpha_1)\Upsilon+\mathcal{O}(\Upsilon^{3/2})$.
The above pill-box array with radial depth d generates a single-frequency resonance impedance at $\omega_r=c\left(\frac{2L}{bgd}\right)^{1/2}$ [16, 17].	$$Z_0^{\parallel}=\frac{Z_0cL}{2\pi b^2}\sum_{\omega'=\pm\omega_r}\left[\pi\delta(\omega-\omega')+\frac{i}{\omega-\omega'}\right]$$ $$Z_1^{\perp}=\frac{2cL}{b^2\omega}Z_0^{\parallel}$$ $$W_0'(z)=\frac{Z_0cL}{\pi b^2}\cos\frac{\omega_r z}{c}$$ $$W_1(z)=\frac{2Z_0L}{\pi b^4\omega_r}\sin\frac{\omega_r z}{c}$$ The corresponding resonator per pill box has $\dfrac{R_s^{(0)}\omega_r}{Q}=\dfrac{Z_0cL}{\pi b^2}$.
Smooth toroidal b and $R=\frac{1}{2}(a+b)$. As the Lorentz factor $\gamma\to\infty$, (ultra-relativistic beam), a <u>curvature contribution</u> remains for the longitudinal impedance [18].	Valid from zero frequency up to just below synchronous resonant modes, i.e., $0<\nu<\sqrt{R/h}$ with $\nu=\omega h/c$, $$\begin{aligned}\frac{Z_0^{\parallel}}{n}&=iZ_0\left(\frac{h}{\pi R}\right)^2\left\{\left[1-e^{-2\pi(b-R)/h}-e^{-2\pi(R-a)/h}\right]\left[1-3\left(\frac{\nu}{\pi}\right)^2\right]\right.\\&\quad\left.+0.05179-0.01355\left(\frac{\nu}{\pi}\right)^2\right\}+\rho\\&\approx iZ_0\left(\frac{h}{\pi R}\right)^2\left[A-3B\left(\frac{\nu}{\pi}\right)^2\right].\end{aligned}$$ where ρ is quadratic in ν. As $(b-a)/h$ increases, ρ vanishes exponentially and $A\approx B\approx 1$. In general, $A/B\approx 1$ implying $\mathrm{Im}Z_0^{\parallel}$ changes sign (a node) near $\nu=\pi/\sqrt{3}$.
<u>Rf cage</u>: beam of radius a surrounded by a cylindrical cage or array of N wires of radius ρ_w, length L at radial distance r_w from beam center. Wire filling factor is $f_w=N\rho_w/(\pi r_w)$. Formulas are valid at low frequencies, $0<n<R/r_w$ and $N\gg 1$.	$$\frac{Z_0^{\parallel}}{n}=\frac{iZ_0L}{4\pi R\beta\gamma^2}\left[1+2\ln\frac{r_w}{a}+C_{\parallel}\right],\quad Z_1^{\perp}=\frac{iZ_0L}{2\pi\beta^2\gamma^2}\left[\frac{1}{a^2}-\frac{1-C_{\perp}}{r_w^2}\right]$$ Without metallic beam pipe outside wire array or cage [19], $$C_{\parallel}=-\frac{2\ln(nr_w/R)\ln(\pi f_w)}{N\ln(nr_w/R)+\ln(\pi f_w)},\quad C_{\perp}=-\frac{2\ln(\pi f_w)}{N-2\ln(\pi f_w)}$$ With infinitely conducting metallic beam pipe, radius $b>r_w$ [20], $$C_{\parallel}=2\ln\frac{b}{r_w}-\frac{2N[\ln(b/r_w)]^2}{N\ln(b/r_w)-\ln(\pi f_w)+\ln[1-(r_w/b)^{2N}]}$$ $$C_{\perp}=\frac{[1-(r_w/b)^2][(r_w/b)^2+(b/r_w)^2]\{\ln[1-(r_w/b)^{2N}]-2\ln(\pi f_w)\}}{N[1-(r_w/b)^2]+[(r_w/b)^2+(b/r_w)^2]\ln[1-(r_w/b)^{2N}]-2\ln(\pi f_w)}$$ A ceramic layer between the wires and metallic beam pipe has negligible effect on the impedances.

Wall roughness [13] 1-D axisymmetric bump, $h(z)$ or 2-D bump $h(z,\theta)$. Valid for low frequency $k=\omega/c \ll$ (bump length or width)$^{-1}$, $h \ll b$, pipe radius, and $\vert\nabla h\vert \ll 1$.	1-D: $Z_0^{\parallel} = -\frac{2ikZ_0}{b}\int_0^{\infty} \kappa \vert\tilde{h}(\kappa)\vert^2 d\kappa$ with spectrum $\tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(z)e^{-ikz}dz$ 2-D: $Z_0^{\parallel} = -\frac{4ikZ_0}{b}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\kappa^2}{\sqrt{\kappa^2+m^2/b^2}}\vert\tilde{h}_m(\kappa)\vert^2 d\kappa$ with spectrum $\tilde{h}_m(k) = \frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dz\, h(z,\theta)e^{-ikz-im\theta}$
Heifets and Kheifets formulas for tapered steps and tapered cavity at high frequencies [14].	
Taper in from radius h to b $(<h)$, out from radius b to h; tapering angle α. Tapering inefficient for a bunch of rms length σ, if $2(h-b)\tan\alpha \gg \sigma$. All formulas here and below are valid for *positive* $k=\omega/c$ only.	$\mathrm{Re}Z_0^{\parallel} = \pm\frac{Z_0}{2\pi}\ln\frac{h}{b} + \left(Z_0^{\parallel}\right)_{\text{step}},\ \mathrm{Re}Z_1^{\perp} = \pm\frac{Z_0 b}{4\pi}\left(\frac{1}{b^2}-\frac{1}{h^2}\right) + \left(Z_1^{\perp}\right)_{\text{step}}\ \begin{cases} +\text{in} \\ -\text{out}\end{cases}$ $\left(Z_0^{\parallel}\right)_{\text{step}} = \frac{Z_0}{2\pi}\ln\frac{h}{b},\ \tan\alpha > \frac{h-b}{kb^2},\quad \left(Z_0^{\parallel}\right)_{\text{step}} = \frac{Z_0}{4}kb\tan\alpha,\ \tan\alpha \ll \frac{1}{kb}$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0}{4\pi b}\left[1-\frac{1}{(1+kb)^2}\,{}_2F_1\left(1,\frac{3}{2},3,\frac{4bh}{(b+h)^2}\right)\right],\ \tan\alpha > \frac{h-b}{kb^2},\ kb \gg 1$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0 b}{4\pi}\left(\frac{1}{b^2}-\frac{1}{h^2}\right),\quad \tan\alpha > \frac{h-b}{kb^2},\ kb\gg 1,\ h \gg b$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0}{16b}(kb)^3\tan\alpha,\quad \tan\alpha \ll \frac{1}{kb}$
Pill-box cavity: total length g, radial depth h without taper. Tapering angle α on both sides, $g \gg h$.	$Z_0^{\parallel} = \begin{cases} \frac{(1+i)Z_0}{2\pi b}\sqrt{\frac{g}{k\pi}} & g \ll kb^2 \\ -i\frac{Z_0}{\pi}\ln\frac{h}{b} & g \gg kb^2 \end{cases}$ $\mathrm{Re}Z_0^{\parallel} = 2\left(Z_0^{\parallel}\right)_{\text{step}},\qquad \mathrm{Re}Z_0^{\perp} = 2\left(Z_0^{\perp}\right)_{\text{step}}$ as given above

1.6 Exercises

1.1. Prove the properties of the impedances in Eqs. (1.31)-(1.34).

1.2. Using a RLC-parallel circuit, derive the longitudinal impedance in Eq. (1.40) by identifying $R_{0s} = R$, $\omega_r = 1/\sqrt{LC}$, and $Q = R\sqrt{C/L}$. Then show that the wake function is $W_0' = 0$ for $z < 0$, and for $z > 0$,

$$W_0'(z) = \frac{\omega_r R_{0s}}{Q} e^{-\alpha z/v} \left[\cos\frac{\bar{\omega}z}{v} - \frac{\alpha}{\bar{\omega}} \sin\frac{\bar{\omega}z}{v}\right] , \tag{1.46}$$

with $\alpha = \omega_r/(2Q)$ and $\bar{\omega} = \sqrt{\omega_r^2 - \alpha^2}$. Similarly, show that

$$W_1(z) = -\frac{R_{1s} v \omega_r}{Q\bar{\omega}_r} e^{-\alpha z/v} \sin\frac{\bar{\omega}z}{v} , \tag{1.47}$$

for $z > 0$ and zero otherwise.

1.3. Show that the wake functions corresponding to the longitudinal resistive wall impedance of Eq. (1.42) and the transverse resistive wall impedance of Eq. (1.44) for a length ℓ are, respectively,

$$W_0'(z) = -\frac{\beta^{3/2} c\ell}{4\pi b z^{3/2}} \sqrt{\frac{Z_0 \mu_r}{\pi\sigma_c}} , \tag{1.48}$$

$$W_1(z) = -\frac{\beta^{3/2} c\ell}{\pi b^3 z^{1/2}} \sqrt{\frac{Z_0 \mu_r}{\pi\sigma_c}} , \tag{1.49}$$

where b is the beam pipe radius, σ_c is the conductivity and μ_r the relative magnetic permeability of the beam pipe walls. The above are only approximates and are valid for $b\chi^{1/3} \ll z \ll b/\chi$, where $\chi = 1/(b\sigma_c Z_0)$. When $z \ll b\chi^{1/3}$, $W_0'(z)$ should have the proper positive sign.

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